Simplifying Calculus

by Using

Uniform Estimates

BY

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Playing with Formulas

Differentiating Polynomials

How to make sense of \( \frac{x^2 - a^2}{x - a} \) for \( x = a \)?

Of course, we just factor the numerator and cancel \( x - a \), so we get

\[
(x^2)' = \frac{x^2 - a^2}{x - a} \bigg|_{x=a} = (x + a)(x - a) \bigg|_{x=a} = 2x,
\]

and now we can differentiate \( x^2 \).

With a bit more work we get

\[
(x^3)' = 3x^2, \quad (x^4)' = 4x^3, \ldots, \quad (x^n)' = nx^{n-1}
\]

This trick will work for any polynomial \( f(x) \)
because \( x - a \) divides \( f(x) - f(a) \), so

\[
f'(x) = \frac{f(x) - f(a)}{x - a} \bigg|_{x=a}
\]

We don’t have to divide polynomials because of...
Differentiation Rules

- \((f + g)' = f' + g'\)
- \((k f)' = kf'\) for any constant \(k\)
- \((f g)' = f'g + fg'\)
- \((f(g(x)))' = f'(g(x))g'(x)\)

Demonstrating these rules for polynomials is a matter of simple algebra of course.

Roots

How to make sense of \(\frac{\sqrt{x} - \sqrt{a}}{x - a}\) for \(x = a\)? It’s the same problem that we started with, turned upside down, so we know what to do.

\[
\left.\frac{\sqrt{x} - \sqrt{a}}{x - a}\right|_{x=a} = \left.\frac{\sqrt{x} - \sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}\right|_{x=a}
\]

so we get \((\sqrt{x})' = \frac{1}{\sqrt{x} + \sqrt{a}}\) \(\big|_{x=a} = \frac{1}{2\sqrt{x}}\)

It’s clear now that \((\sqrt[n]{x})' = \frac{1}{n(\sqrt[n]{x})^{n-1}}\)

(powers upside down, again)
Another way to derive the formula for \((\sqrt[n]{x})'\) is to rewrite \(y = \sqrt[n]{x}\) as \(y^n = x\), to differentiate this equation to get \(n y^{n-1} y' = 1\) and to solve for \(y'\). This trick, called *implicit differentiation*, makes it easy to get \((x^{m/n})'\), \((u/v)'\) and even \(y'\) if \(y^7 + y + x = 0\), when we are at a loss to derive a formula for \(y\) itself.

We are stretching it a bit here, of course, by assuming that \(y'\) is defined, but it turns out O.K. if we don’t have to divide by zero, as the *implicit function theorem* says.

**An Application: a Holy Bucket.**

![Diagram](image)

From energy conservation \(v = \sqrt{2gH}\),
from incompressibility \(AH' = -eav\),
where \(e\) is the *efflux* coefficient.
sin and cos

$|OB|=|BD|=1$

$|CD|=|OA|$
implies $\sin t = \cos t$

$|CB|=|AB|$
implies $\cos t = -\sin t$
Areas, Newton-Leibniz by example

\[ A(x) = x, \text{ so } A'(x) = 1. \]

\[ A(x) = x^2/2, \text{ so } A'(x) = x. \]
What about the other powers? Fermat’s idea:

\[ A(B) = \frac{B^{k+1}}{k+1}, \text{ so } A'(B) = B^k \]
Archimedes approach:

\[ y = ax^2 \]
Uniform grid approach:

**Antiderivatives and integrals**

\[ F' = f \iff \int f(x)\,dx = F(x) + C \]

\[ \int x^k\,dx = \frac{x^{k+1}}{k+1} + C \text{ for } k \neq -1, \]

\[ \int \cos = \sin + C, \quad \int \sin = -\cos + C, \text{ etc.} \]

\[ \int_a^b f(x)\,dx = F(b) - F(a), \quad f = F' \]
Integration rules, positivity, additivity.

Now what about \( \int \frac{dx}{x} \)?

\[
\int_{a}^{b} \frac{dx}{x} = \frac{b^{0} - a^{0}}{-1 + 1} = \frac{0}{0},
\]

and we meet our old friend again.

But geometrically speaking, the area under \(1/x\) makes sense, we just have to figure out what it is. To do it, we just look at the picture...

...and see that it is some sort of a logarithm. It is called the natural logarithm, so

\[
\int_{a}^{b} \frac{dx}{x} = \ln(b) - \ln(a), \quad \int \frac{dx}{x} = \ln(x) + C,
\]

and \((e^{x})' = e^{x}\) (by implicit differentiation).
Playing with Inequalities

Why a tangent looks like a tangent

After examining a few examples

we arrive at the estimate

\[ |f(x) - f(a) - f'(a)(x - a)| \leq K(x - a)^2 \]

and call \( f \) ULD (uniformly Lipschitz differentiable).

It follows that

\[ \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| \leq K|x - a|, \]

and we conclude that \( |f'(x) - f'(a)| \leq 2K|x - a| \),

i.e. \( f' \) is Lipschitz.
Increasing function theorem

\[ f' \geq 0 \text{ and } A \leq B \Rightarrow f(A) \leq f(B) \]

We first assume that \( f' \geq c > 0 \) and look at the estimate defining ULD. We see that
\[ 0 \leq x - a \leq c/K \Rightarrow f(a) \leq f(x), \]
and therefore \( f(A) \leq f(B) \) because we can get from \( A \) to \( B \) by taking steps shorter than \( c/K \) (according to Archimedes).

Now for \( f' \geq 0 \) we can conclude that
\[ f(B) - f(A) \geq -c(B - A) \text{ for any } c > 0, \]
and therefore \( f(A) \leq f(B) \). Q.E.D.
Integrability of Lipschitz functions and Newton-Leibniz

We can pick piecewise-linear \( \bar{f} \) and \( \tilde{f} \),
\[
\underline{f} \leq f \leq \tilde{f} \quad \text{and} \quad \tilde{f} - \underline{f} \leq 4Lh,
\]
where \( h \) is the mesh size and \( L \) is the Lipschitz constant for \( f \). Then the inequality

\[
\int_a^b \underline{f} \leq \int_a^b f \leq \int_a^b \tilde{f}
\]

will define \( \int_a^b f \) uniquely. Positivity, additivity and Newton-Leibniz follow.