Simplifying Calculus

by Using

Uniform Estimates

BY

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Playing with Formulas

Differentiating Polynomials

How to make sense of $\dfrac{x^2 - a^2}{x - a}$ for $x = a$?

Of course, we just factor the numerator
and cancel $x - a$, so we get

$$(x^2)' = \frac{x^2 - a^2}{x - a}\,|_{x=a} = \frac{(x+a)(x-a)}{x-a}|_{x=a} = 2x\,,$$

and now we can differentiate x^2.
With a bit more work we get

$$(x^3)' = 3x^2, (x^4)' = 4x^3, \ldots, (x^n)' = n\,x^{n-1}$$

This trick will work for any polynomial $f(x)$
because $x - a$ divides $f(x) - f(a)$, so

$$f'(x) = \frac{f(x) - f(a)}{x - a}|_{x=a}$$

We don't have to divide polynomials because of...

Differentiation Rules

- $(f+g)' = f' + g'$
- $(kf)' = kf'$ for any constant k
- $(fg)' = f'g + fg'$
- $(f(g(x))' = f'(g(x))g'(x)$

Demonstrating these rules for polynomials
is a matter of simple algebra of course.

Roots

How to make sense of $\frac{\sqrt{x}-\sqrt{a}}{x-a}$ for $x = a$?

It's the same problem that we started with,
turned upside down, so we know what to do.

$$\frac{\sqrt{x}-\sqrt{a}}{x-a}\,|_{x=a} = \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}|_{x=a}$$

so we get $(\sqrt{x})' = \frac{1}{\sqrt{x}+\sqrt{a}}|_{x=a} = \frac{1}{2\sqrt{x}}$

It's clear now that $(\sqrt[n]{x})' = \frac{1}{n(\sqrt[n]{x})^{n-1}}$

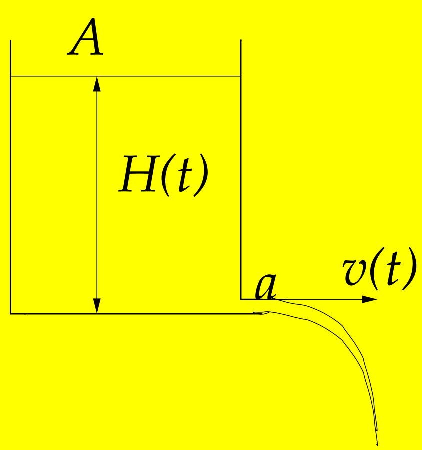
(powers upside down, again)

Implicit Differentiation, Quotients

Another way to derive the formula for $(\sqrt[n]{x})'$ is to rewrite $y = \sqrt[n]{x}$ as $y^n = x$, to differentiate this equation to get $ny^{n-1}y' = 1$ and to solve for y'. This trick, called *implicit differentiation*, makes it easy to get $(x^{m/n})'$, $(u/v)'$ and even y' if $y^7 + y + x = 0$, when we are at a loss to derive a formula for y itself.

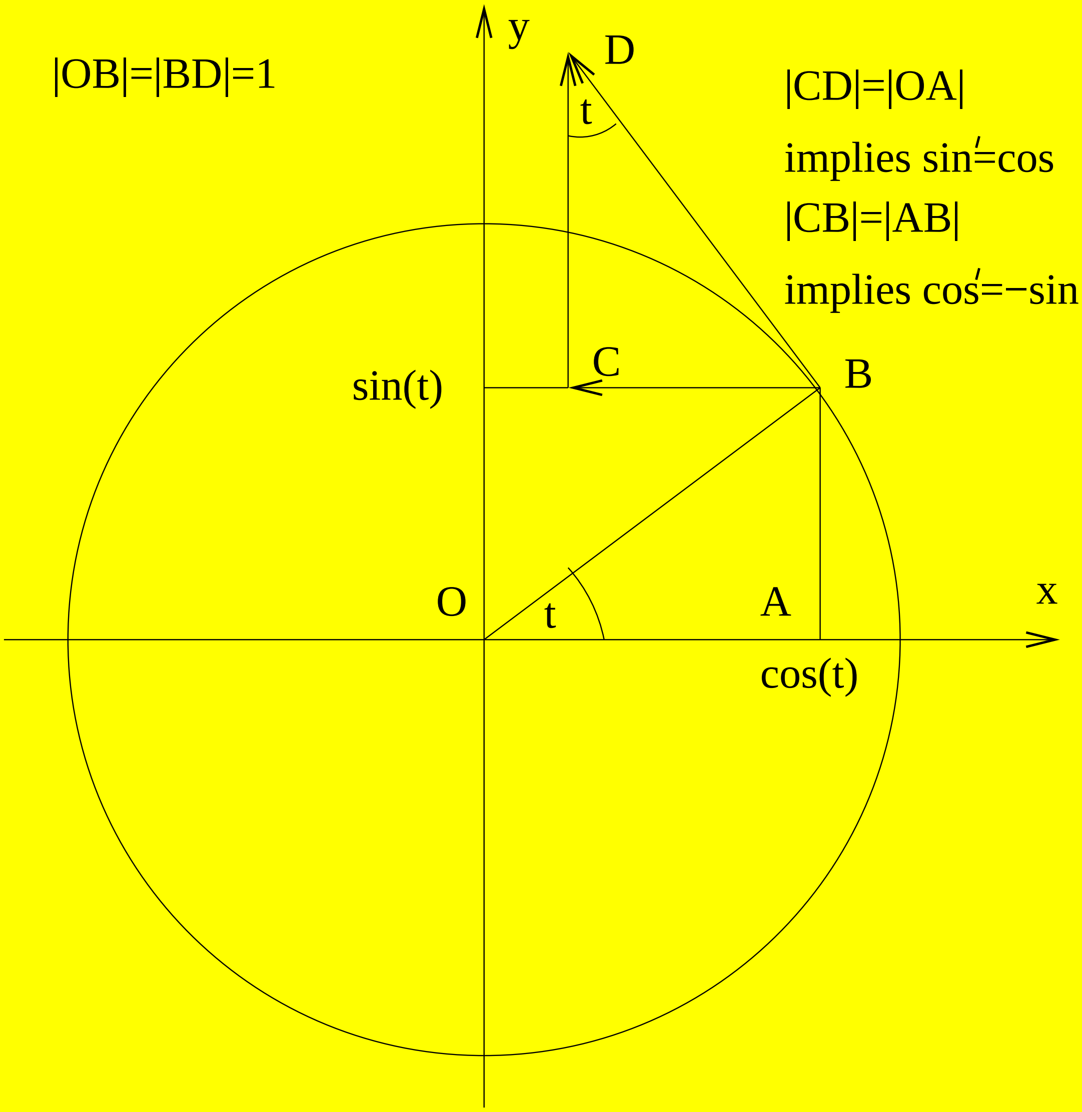
We are stretching it a bit here, of course, by assuming that y' is defined, but it turnes out O.K. if we don't have to divide by zero, as the *implicit function theorem* says.

An Application: a Holy Bucket.

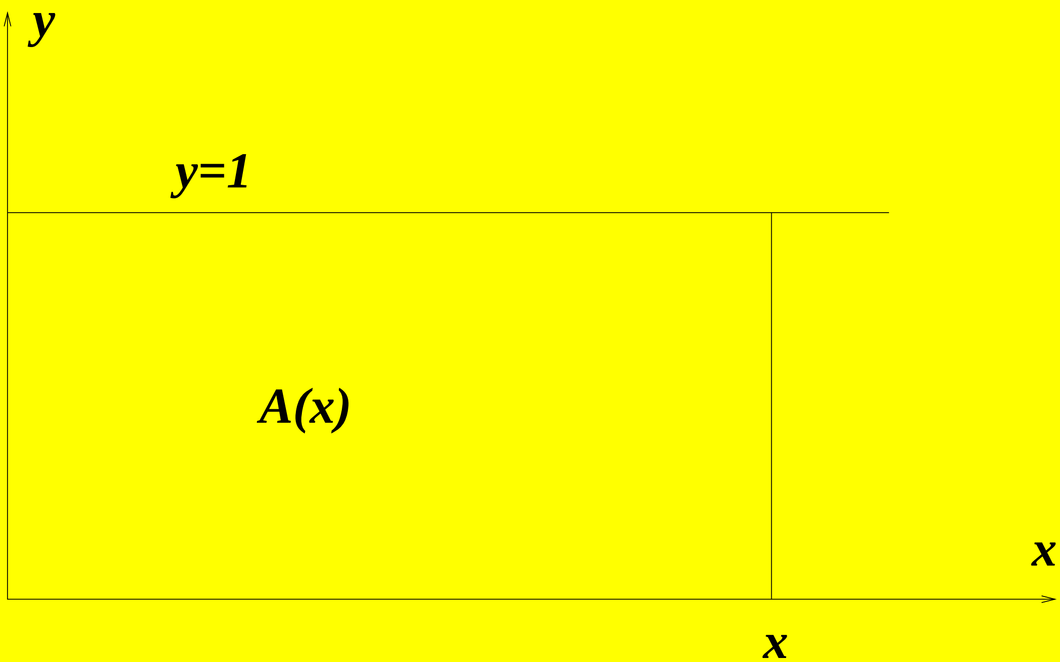


From energy conservation $v = \sqrt{2gH}$, from incompressibility $AH' = -eav$, where e is the *efflux* coefficient.

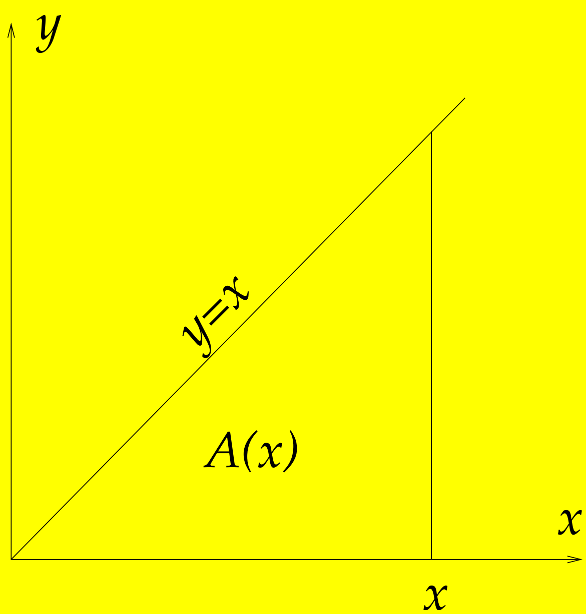
sin and cos


|OB|=|BD|=1
y
D
t
|CD|=|OA|
implies sin′=cos
|CB|=|AB|
implies cos′=−sin
sin(t)
C
B
O
t
A
x
cos(t)

Areas, Newton-Leibniz by example

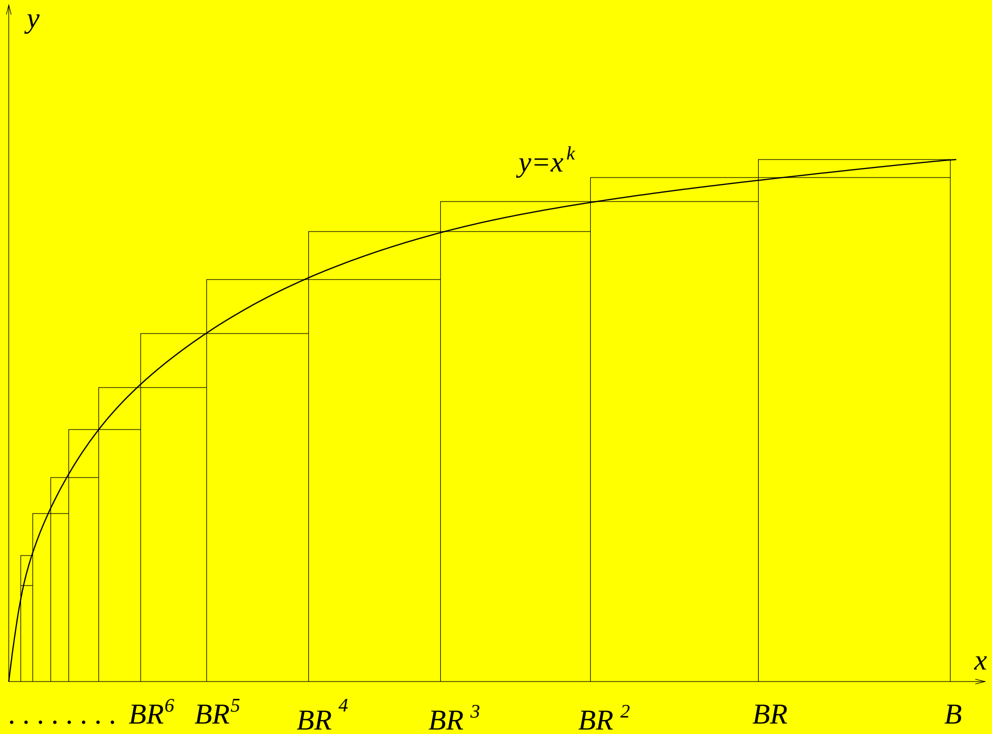


$$A(x) = x, \text{ so } A'(x) = 1.$$



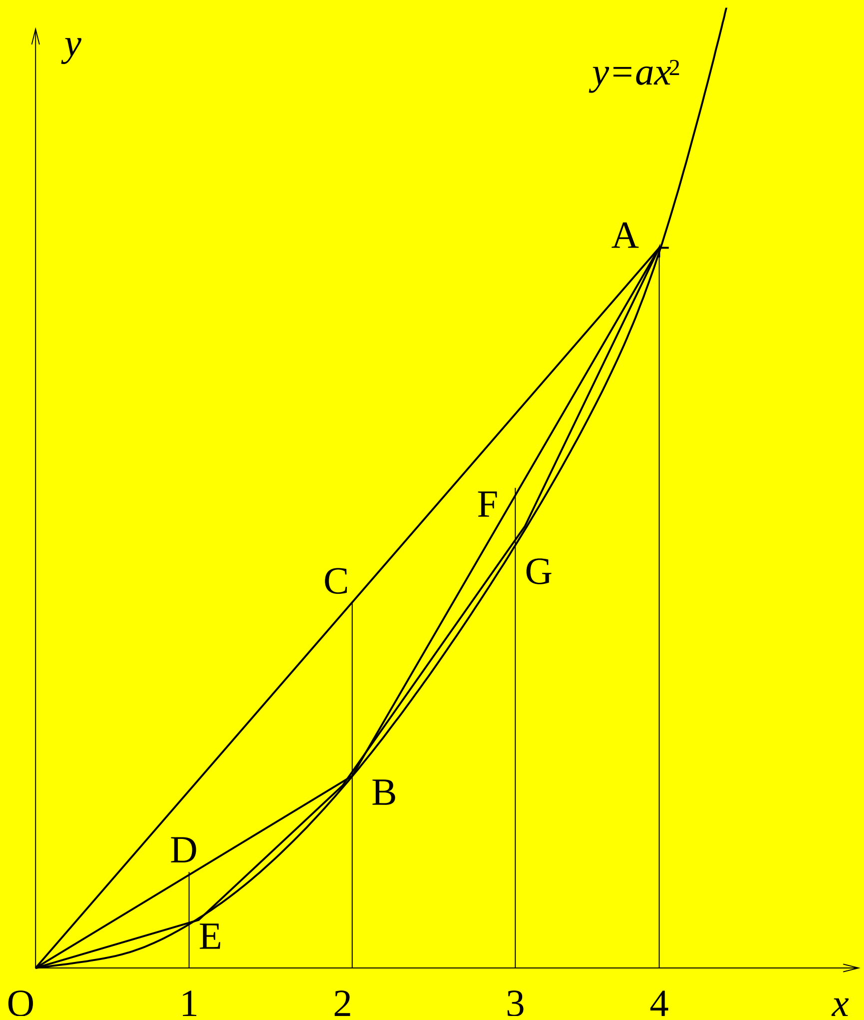
$$A(x) = x^2/2, \text{so } A'(x) = x.$$

What about the other powers? Fermat's idea:

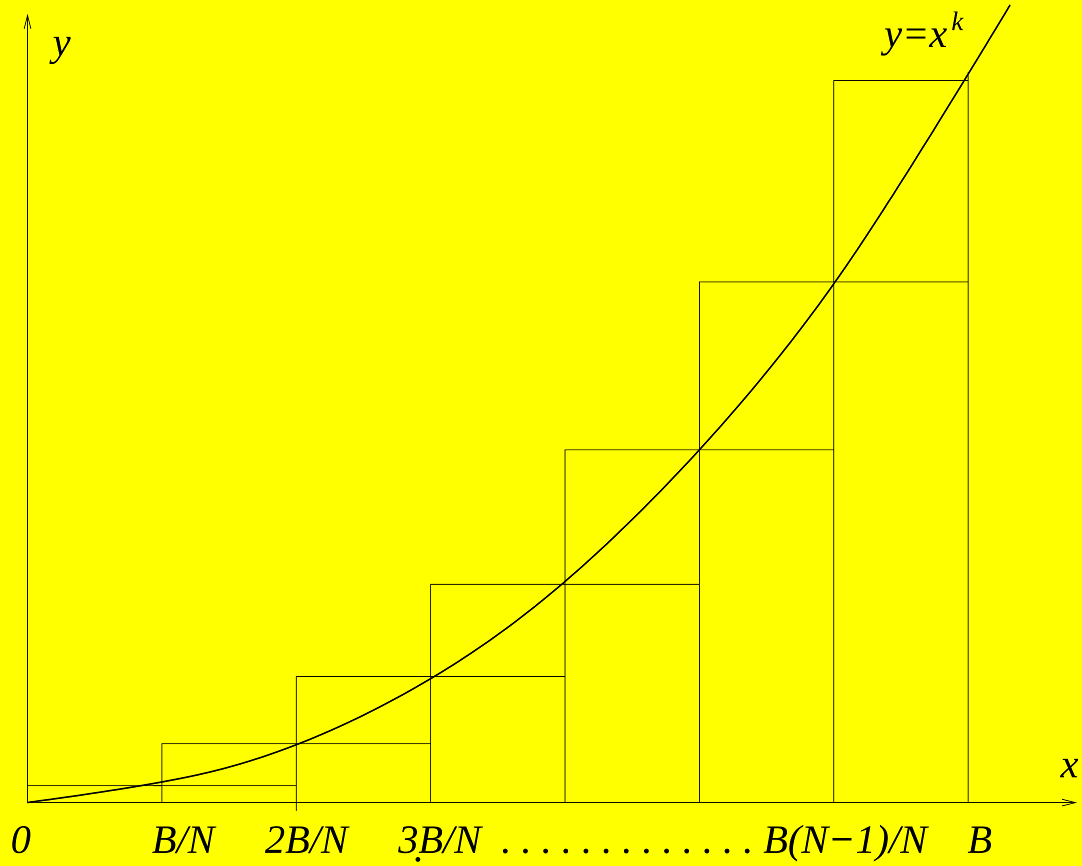


$$A(B) = \frac{B^{k+1}}{k+1}\ ,\ \text{so}\ A'(B) = B^k$$

Archimedes approach:



Uniform grid approach:



Antiderivatives and integrals

$$F' = f \Leftrightarrow \int f(x)\,dx = F(x) + C$$

$$\int x^k dx = x^{k+1}/(k+1) + C \text{ for } k \neq -1,$$

$$\int cos = sin + C, \ \int sin = -cos + C, \text{ etc.}$$

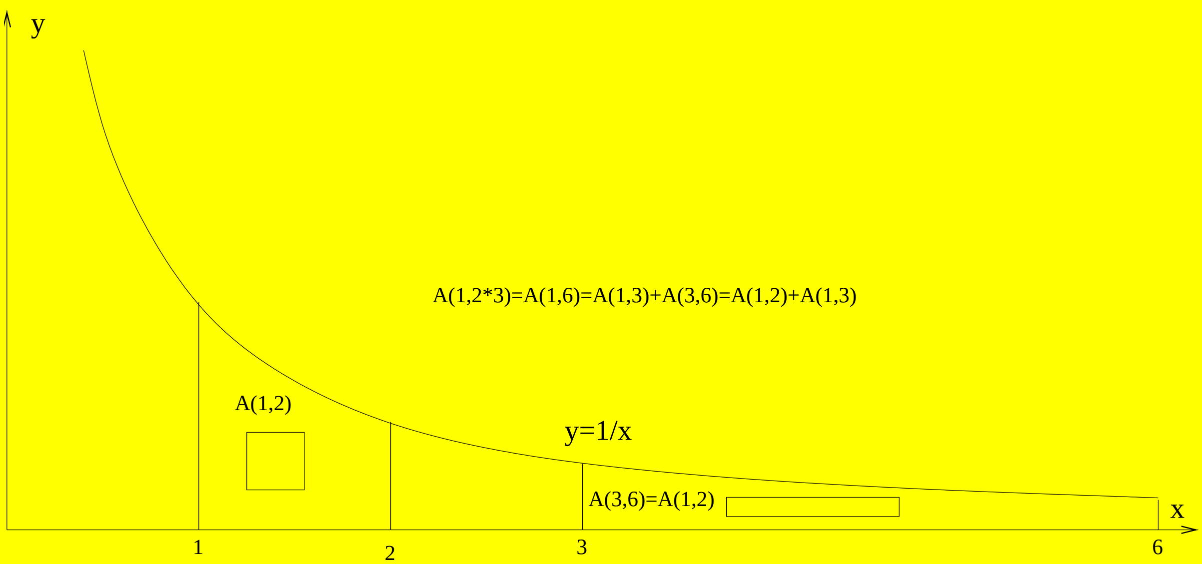
$$\int_a^b f(x)dx = F(b) - F(a), \ f = F'$$

Now what about $\int\ dx/x$?

$$\int_a^b dx/x = \frac{b^0 - a^0}{-1+1} = \frac{0}{0},$$

and we meet our old friend again.

But geometrically speaking, the area under $1/x$ makes sense, we just have to figure out what it is. To do it, we just look at the picture...



...and see that it is some sort of a logarithm. It is called the natural logarithm, so

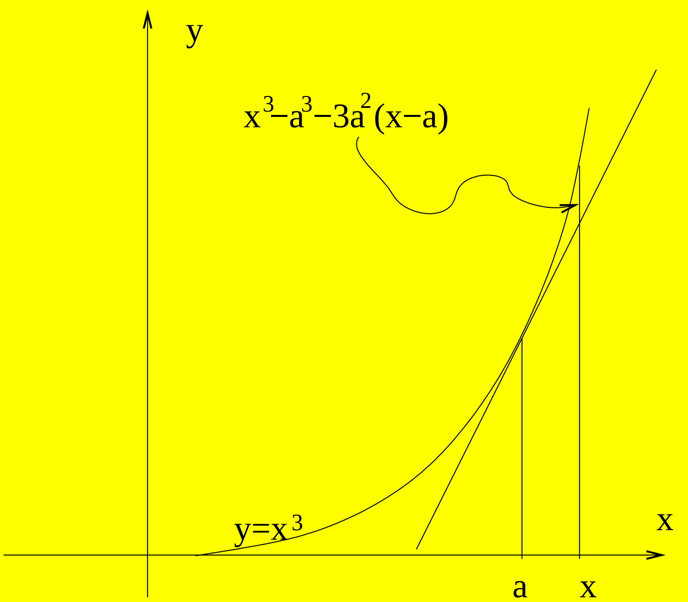
$$\int_a^b dx/x = ln(b) - ln(a),\ \int dx/x = ln(x) + C,$$

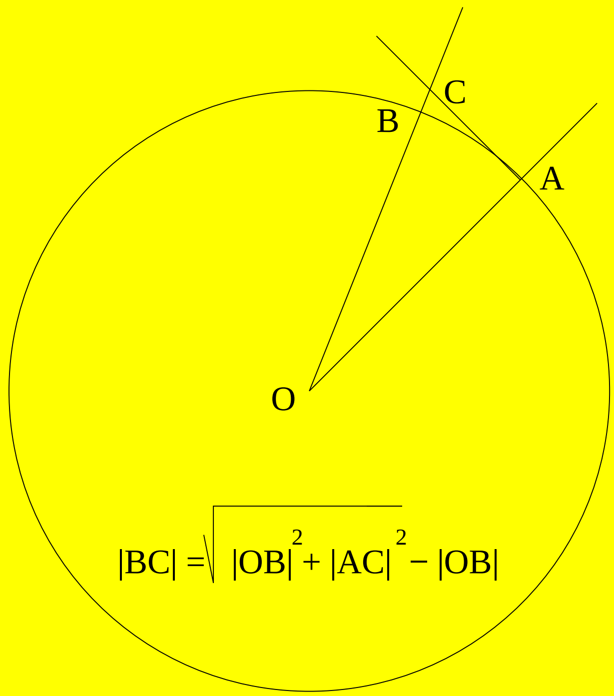
and $(e^x)' = e^x$ (by implicit differentiation).

Playing with Inequalities

Why a tangent looks like a tangent

After examining a few examples





we arrive at the estimate

$$\left|f(x) - f(a) - f'(a)(x-a)\right| \leqslant K(x-a)^2$$

and call f ULD (uniformly Lipschitz differentiable).

It follows that

$$\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| \leqslant K\,|x-a|,$$

and we conclude that $|f'(x) - f'(a)| \leqslant 2K\,|x-a|$,
i.e. f' is Lipschitz.

Increasing function theorem

$$f' \geqslant 0 \text{ and } A \leqslant B \Rightarrow f(A) \leqslant f(B)$$

We first assume that $f' \geqslant c > 0$ and look at the estimate defining ULD. We see that

$$0 \leqslant x - a \leqslant c/K \Rightarrow f(a) \leqslant f(x),$$

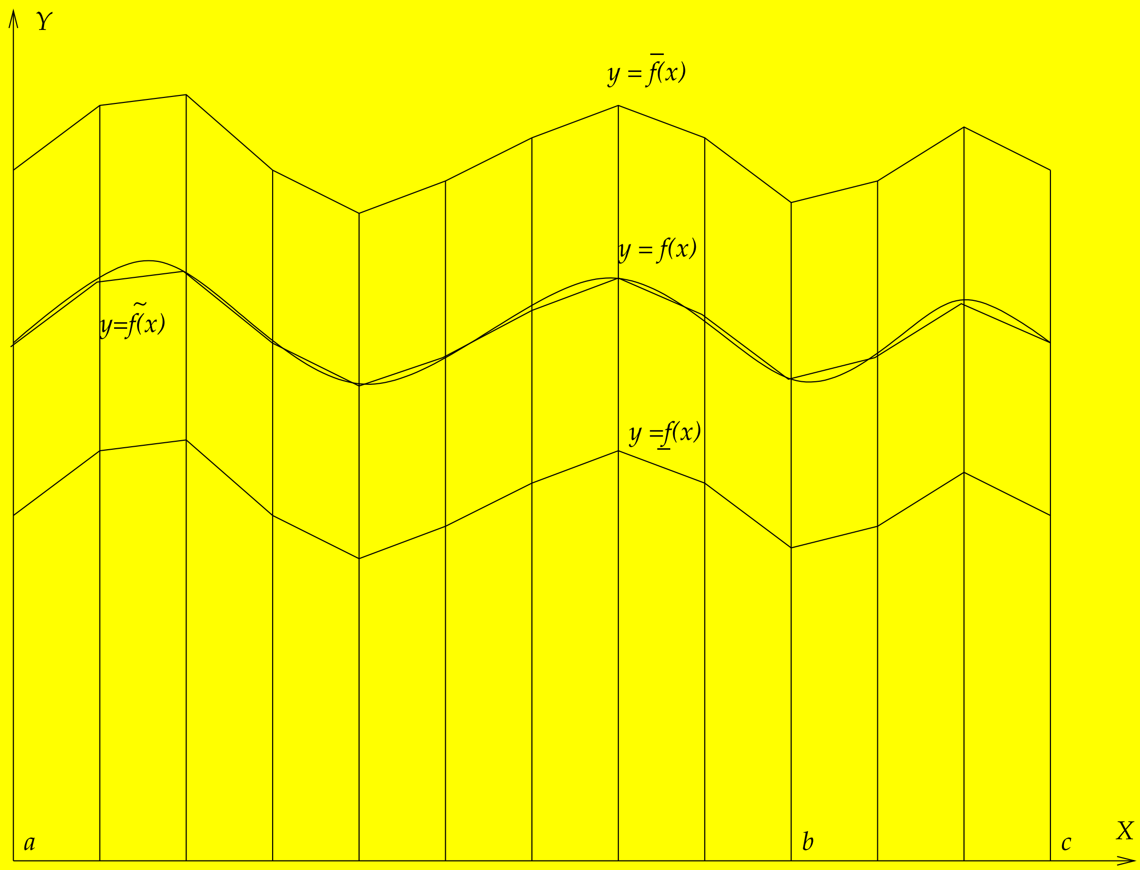
and therefore $f(A) \leqslant f(B)$ because we can get from A to B by taking steps shorter than c/K (according to Archimedes).

Now for $f' \geqslant 0$ we can conclude that

$$f(B) - f(A) \geqslant -c(B - A) \text{ for any } c > 0,$$

and therefore $f(A) \leqslant f(B)$. Q.E.D.

Integrability of Lipschitz functions and Newton-Leibniz



We can pick piecewise – linear $\bar{f}$ and $\underline{f}$,

$\underline{f} \leqslant f \leqslant \bar{f}$ and $\bar{f} - \underline{f} \leqslant 4Lh$, where h is

the mesh size and L is the Lipschitz constant for f. Then the inequality

$$\int_a^b \underline{f} \leqslant \int_a^b f \leqslant \int_a^b \bar{f}$$

will define $\int_a^b f$ uniquely. Positivity, additivity and Newton-Leibniz follow.